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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

MSC INTERNAL NOTE NO. 67-FM-83

June 23, 1967

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SUMMARY AND INTRODUCTION

This report presents the results of a study to determine a minimum ΔV, two-impulse transfer trajectory from a 50-n. mi. circular parking orbit above the planet Mars onto a coplanar escape hyperbola that has been arbitrarily oriented and has an eccentricity of 7.46574006 and a semilatus rectum of 5.02191087 e.r. This study was conducted utilizing the accelerated gradient method program (ref. 1) which in most cases was modified to simultaneously determine both (1) a best insertion point onto the hyperbolic escape trajectory and (2) the minimum ΔV two-impulse transfer onto the hyperbolic escape trajectory at this point. This modification assumes that there are given fixed initial position and velocity vectors in the orbital plane of the target conic.

SYMBOLS

a	semimajor axis of target hyperbola, e.r.
e	eccentricity of target hyperbola, n.d.
F	performance index, e.r./hr
F'	$F + \lambda^T g$
g	terminal constraints
h	scalar angular momentum of target hyperbola, $(\text{e.r.})^2/\text{hr}$
p	semilatus rectum of target hyperbola, e.r.
r	$(x^2 + y^2)^{1/2}$, e.r.
x, y, u^+, v^+	components of state vector after terminal impuse, e.r., e.r., e.r./hr, e.r./hr

X, Y	preassigned rectangular coordinate axes
α	control vector
α^*	value of control vector for which $\frac{\partial F'}{\partial \alpha}$ vector vanishes
β	change in generalized eccentric anomaly, $(\text{e.r.})^{1/2}$
$\Delta V_o = \begin{bmatrix} \Delta u_o \\ \Delta v_o \end{bmatrix}$	initial velocity impulse vector, e.r./hr
$\Delta V_f = \begin{bmatrix} \Delta u_f \\ \Delta v_f \end{bmatrix}$	final velocity impulse vector, e.r./hr
θ_f	polar angle measured positively from periapsis in the direction of motion, specifying the insertion point, deg
θ_o	true anomaly angle for first impulse maneuver, deg
$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$	vector of constant Lagrange multipliers
λ^T	transpose of λ
μ	Mars gravitational constant, 2.146659 $(\text{e.r.})^3/\text{hr}^2$
ϕ	inclination of periapsis of target hyperbola relative to +X axis, deg

METHOD

In each computer run generated in the study a transfer trajectory was obtained by determining control vector α^* which minimizes the performance index

$$F(\alpha) = |\Delta V_o| + |\Delta V_f|$$

subject to the terminal constraints

$$g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} r + ex\cos\phi + ey\sin\phi - p \\ u^+ + \frac{hy}{pr} + \frac{he\sin\phi}{p} \\ v^+ - \frac{hx}{pr} - \frac{he\cos\phi}{p} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

derived in the appendix. The control vector α is defined by

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} \Delta u_f \\ \Delta v_f \\ \Delta u_o \\ \Delta v_o \\ \beta \end{bmatrix}$$

where $(\Delta u_o, \Delta v_o)$ and $(\Delta u_f, \Delta v_f)$ are the rectangular components of ΔV_o and ΔV_f, respectively, and β is the corresponding change in "generalized eccentric anomaly" (ref. 2), i.e., the independent variable associated with universal two-body equations and partial derivatives.

The target hyperbola is oriented such that $\phi = 0°$(i.e., periapsis is on the positive X-axis), e = 7.46574006, and p = 5.02191087 e.r. Further, by $h = \sqrt{\mu p}$ and $\mu = 2.146659\ (\text{e.r.})^3/\text{hr}^2$, $h = 3.28334\ (\text{e.r.})^2/\text{hr}$. The corresponding periapsis altitude and velocity on the target hyperbola were 0.59320400 e.r. and 5.5349278 e.r./hr. The characteristic scalar velocity on the 50-n. mi. circular parking orbit of radius 0.54964918 e.r. prior to the transfer maneuver was 1.9762357 e.r./hr.

Verification of a relative minimum for F was obtained by comparing to zero the values found for the components of g and $\partial F'/\partial\alpha$, where $F' = F + \lambda^T g$.

Thus,

$$\frac{\partial F'}{\partial \alpha} = \begin{bmatrix} \frac{\partial F'}{\partial \alpha_1} \\ \frac{\partial F'}{\partial \alpha_2} \\ \frac{\partial F'}{\partial \alpha_3} \\ \frac{\partial F'}{\partial \alpha_4} \\ \frac{\partial F'}{\partial \alpha_5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is a necessary condition for a minimum of F subject to the constraints $g = 0$.

In certain exceptional cases convergence could not be obtained because of numerical difficulties arising when the optimization program was tending to select an insertion point well beyond the hyperbolic periapsis. Here the problem was replaced by an approximately equivalent problem of minimizing the performance index F subject to terminal constraints characterizing a specified state vector either 10 or 15 hours of coasting flight time beyond the hyperbolic periapsis; i.e., subject to

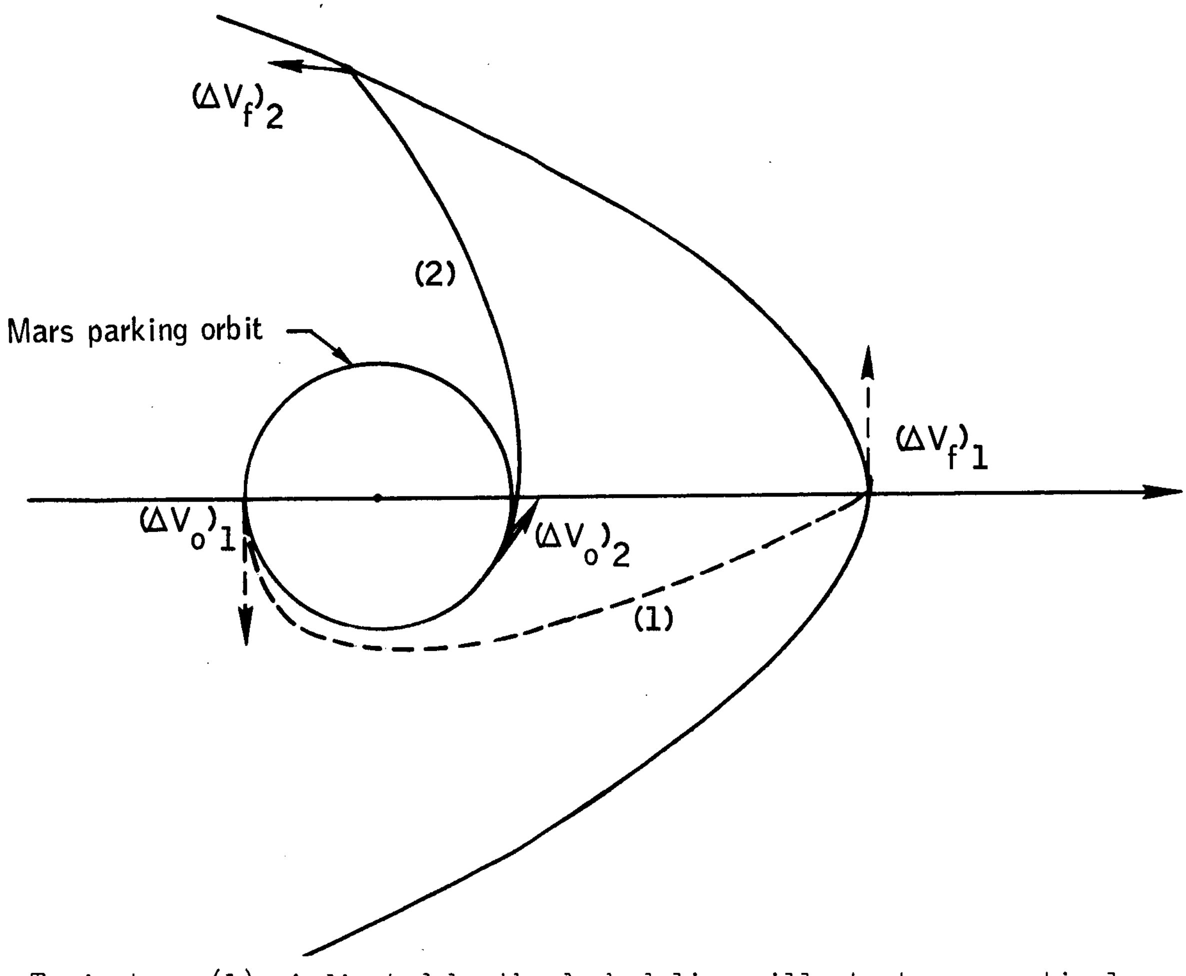
$$g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} x + 5.8557611 \text{ e.r.} \\ y - 48.3864552 \text{ e.r.} \\ u^+ + 0.6490673 \text{ e.r./hr} \\ v^+ - 4.8025740 \text{ e.r./hr} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for the 10-hour case, or to

$$g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} x + 8.9919391 \text{ e.r.} \\ y - 71.5908990 \text{ e.r.} \\ u^+ + 0.6487062 \text{ e.r./hr} \\ v^+ - 4.7996360 \text{ e.r./hr} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for the 15-hour case.

The study consisted of a sequence of runs assuming an impulse-coast-impulse type maneuver and using the accelerated gradient method program. For each case initial position and velocity components on the Mars circular parking orbit were input to the program. First guesses for α were made to start the program coverging to find α^*. The first impulse, ΔV_o, for the transfer was made at true anomaly θ_o on the parking orbit, and the second impulse, ΔV_f, at the polar angle, θ_f, which is the point of insertion onto the hyperbolic trajectory. Two typical cases are shown in the following figure.



Trajectory (1), indicated by the dashed line, illustrates a particular case; namely, $\theta_o = -180°$, for which the insertion point onto the hyperbola is optimized by the program. Here the program selects $\theta_f = 0°$, i.e., a Hohmann transfer. The incremental velocity impulses, $(\Delta V_o)_1$ and $(\Delta V_f)_1$, for this elliptical transfer trajectory are illustrated by the dashed arrows. Trajectory (2), indicated by the solid line, illustrates a particular case for which the insertion point is prespecified, namely $\theta_o = -0.5°$, $\theta_f = 96.9004°$. The incremental velocity impulses $(\Delta V_o)_2$ and $(\Delta V_f)_2$ for this hyperbolic transfer trajectory are illustrated by the solid arrows.

RESULTS

The results are given in the table, which gives polar angle θ_f for insertion, performance index F, and control vector α^* for each case. The maneuver resulting in the lowest performance index obtained was given by run no. 14, a two-impulse transfer initiated at the point on the circular parking orbit coincident with the line of symmetry of the target hyperbola and terminated at a fixed point lying 15 hours of flight time beyond the periapsis. Here, the control vector selected is given by

$$\alpha^* = \begin{bmatrix} 0.00360 & \text{e.r./hr} \\ 0.00469 & \text{e.r./hr} \\ 0.04912 & \text{e.r./hr} \\ 3.6056 & \text{e.r./hr} \\ 1.6399 & (\text{e.r.})^{1/2} \end{bmatrix}$$

and the corresponding minimum performance index was found to be $F(\alpha^*) = 3.6118429$ e.r./hr. The constraints for run no. 14 were well satisfied, thus

$$g = \begin{bmatrix} -0.178 \times 10^{-14} & \text{e.r.} \\ +0.284 \times 10^{-14} & \text{e.r.} \\ +0.555 \times 10^{-16} & \text{e.r./hr} \\ +0.144 \times 10^{-15} & \text{e.r./hr} \end{bmatrix}$$

Also, the necessary condition for a minimum of F subject to the constraints $g = 0$, namely $\partial F'/\partial\alpha = 0$, was well satisfied, for there was found

$$\frac{\partial F'}{\partial \alpha} = \begin{bmatrix} -0.579 \times 10^{-13} \\ -0.505 \times 10^{-13} \\ 0.349 \times 10^{-12} \\ 0.168 \times 10^{-11} \\ -0.429 \times 10^{-11} \end{bmatrix}$$

CONCLUDING REMARKS

It should be noted that for several runs attempted permitting a variable initial coast (as well as two-impulses) numerical difficulties were encountered. This might be interpreted as due to the relative lack of dependence of the optimum performance index on the position in the parking orbit at which the transfer is initiated. This lack of dependence may be regarded as due to the relative proximity of the hyperbolic periapsis to the parking orbit; it is known that for circular cotangency with hyperbolic periapsis the optimum two-impulse transfer degenerates to a one-impulse maneuver and hence is totally independent of θ_o. A further run which failed to attain complete convergence indicated that the absolute optimum two-impulse transfer might be the lower bound $\theta_o = -0.533^{\circ}$, $\theta_f = 97.700^{\circ}$, and $F \approx 3.28$. This indicates that the point of insertion approaches infinity on the hyperbolic asymptote as the second impulse tends to vanish. This result is supported by the analysis in reference 3.

TABLE.- RESULTS OF OPTIMIZING THE TRANSFER FROM MARS PARKING ORBIT TO A HYPERBOLIC ESCAPE TRAJECTORY

Run no.	Initial position on circle, θ_o, deg	Target	Eccentricity of transfer conic, n.d.	Orientation of insertion point, θ_f, deg	Order of magnitude of $\frac{\partial F'}{\partial \alpha}$	Order of magnitude of constraints	F, e.r./hr	α_1^*, e.r./hr	α_2^*, e.r./hr	α_3^*, e.r./hr	α_4^*, e.r./hr	α_5^*, (e.r.)$^{1/2}$
1	-300	Any point on hyperbola	0.1518	-5.7536	10^{-11}	10^{-14}	3.7062930	0.12285	3.5595	-0.12516	+0.07260	-0.81527
2	-270	Any point on hyperbola	1.0000	-3.3333	10^{-10}	10^{-13}	3.7064524	0.07245	3.6319	-0.07384	+0.00000	3.5072
3	-180	Any point on hyperbola	0.0381	0.0000	10^{-13}	10^{-15}	3.7065324	0.00000	3.6692	-0.00000	-0.0373	2.3291
4	-135	Any point on hyperbola	0.5261	1.3833	10^{-13}	10^{-15}	3.7065174	-0.03028	3.6628	-0.03086	-0.03083	1.7414
5	-90	Any point on hyperbola	0.0761	3.3248	10^{-11}	10^{-13}	3.7064524	-0.07245	3.6319	0.00008	1.1975	0.07384
6	-90	Vertex	0.0760	0.0000	10^{-14}	10^{-15}	3.7123099	-0.14463	3.6356	0.07364	-0.00588	1.1550
7	-45	Any point on hyperbola	0.2579	8.0111	10^{-11}	10^{-15}	3.7060685	-0.16633	3.4619	9.16933	0.17037	0.62955
8	-22.5	Any point on hyperbola	0.9500	16.6412	10^{-12}	10^{-15}	3.7045440	-0.28965	2.9067	0.29302	0.72657	0.3767
9	-21	Any point on hyperbola	1.0787	17.8704	10^{-12}	10^{-15}	3.7042450	-0.30114	2.8151	0.30413	0.81836	0.3630
10	-20	Any point on hyperbola	1.1793	18.7961	10^{-11}	10^{-14}	3.7040071	-0.30883	2.7455	0.31146	0.88820	0.3544

TABLE.- RESULTS OF OPTIMIZING THE TRANSFER FROM MARS PARKING ORBIT TO A HYPERBOLIC ESCAPE TRAJECTORY - Concluded.

Run no.	Initial position on circle, θ_o, deg	Target	Eccentricity of transfer conic, n.d.	Orientation of insertion point, θ_f, deg	Order of magnitude of $\frac{\partial F'}{\partial \alpha}$	Order of magnitude of constraints	F, e.r./hr	α_1^*, e.r./hr	α_2^*, e.r./hr	α_3^*, e.r./hr	α_4^*, e.r./hr	α_5^*, (e.r.)$^{1/2}$
11	-0.5	Point 10 hours beyond vertex	6.9721	96.9004	10^{-11}	10^{-13}	3.6124224	-0.00531	0.00696	0.03996	3.6034	1.5240
12	0	Point 10 hours beyond vertex	6.9719	96.9004	10^{-11}	10^{-13}	3.6125485	-0.00531	0.00693	0.05113	3.6035	1.5217
13	0.5	Point 10 hours beyond vertex	6.9713	96.9004	10^{-12}	10^{-14}	3.6128554	-0.00532	0.00690	0.06233	3.6036	1.5195
14	0	Point 15 hours beyond vertex	6.9780	97.1589	10^{-11}	10^{-13}	3.6118429	-0.00360	0.00469	0.00469	3.6056	1.6399

APPENDIX

DERIVATION OF TERMINAL CONSTRAINTS CHARACTERIZING A GIVEN CONIC

The three constraints which require that the terminal state vector characterize a conic with specified semi-latus rectum (p), eccentricity (e), and inclination (ϕ) are herein derived. First, the terminal position coordinates must satisfy the polar equation for a conic,

$$r = \frac{p}{1 + e \cos (\theta - \phi)} \tag{1}$$

By appropriate use of the transformation equations $x = r \cos \theta$, $y = r \sin \theta$ and the identity $\cos (\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi$, it is seen that the terminal rectangular position coordinates x, y must satisfy $g_1 = 0$, where

$$g_1 = r + ex \cos \phi + ey \sin \phi - p \tag{2}$$

where r is given by $r = \sqrt{x^2 + y^2}$.

Next, the terminal velocity components in rectangular coordinates are determined. By differentiation of (2) there follows

$$\dot{r} = e\dot{x} \cos \phi - e\dot{y} \sin \phi \tag{3}$$

But by differentiation of $r = \sqrt{x^2 + y^2}$ there follows

$$r\dot{r} = x\dot{x} + y\dot{y} \tag{4}$$

By (3) and (4) there is obtained

$$(x + re \cos \phi)\dot{x} + (y + re \sin \phi)\dot{y} = 0 \tag{5}$$

This result is compared to an altered version of the equation obtained by differentiation of (1), i.e.,

$$\dot{r} = \frac{pe \sin (\theta - \phi)\dot{\theta}}{[1 + e \cos (\theta - \phi)]^2} \tag{6}$$

Equation (6) is first altered in form by use of equation (1). Then the Kepler equation

$$\dot{\theta} = \frac{h}{r^2} \tag{7}$$

where $h = \sqrt{\mu p}$ and μ is the gravitational constant, is substituted into (6), yielding

$$\dot{r} = \frac{eh}{p} \sin (\theta - \phi) \tag{8}$$

Finally, equation (8) is altered by use of equation (4), the identity $\sin (\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi$, and the transformation equations $x = r \cos \theta$, $y = r \sin \theta$ to yield

$$x\dot{x} + y\dot{y} = \frac{yeh \cos \phi}{p} - \frac{xeh \sin \phi}{p} \tag{9}$$

By simultaneous solution of (5) and (9) it is found that the terminal rectangular velocity components $\dot{x}$, $\dot{y}$ must be related to the terminal rectangular position coordinates x, y by $g_2 = 0$ and $g_3 = 0$, respectively, where

$$g_2 = \dot{x} + \frac{hy}{pr} + \frac{he \sin \phi}{p} \tag{10}$$

and

$$g_3 = \dot{y} - \frac{hx}{pr} - \frac{he \cos \phi}{p} \tag{11}$$

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